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Normal ordering of the Dirac radial momentum operator and the power of radial coordinate operators by virtue of the IWOP technique*

Hongyi Fan[1,2] **and Junhua Chen**[3]

[1] CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China
[2] Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China
[3] Special Class of Gifted Young, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

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Abstract
By virtue of the technique of integration within an ordered product of operators we derive the normal ordering expansion of the Dirac's radial momentum operator. To realize this goal, we also derive some operator identities of the power of radial coordinate operators. They are useful in calculating expectation values in the coherent state.

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1. Introduction

It is well known that normally ordered expansions of operators are very useful in handling miscellaneous calculations of expectation values of observables in the coherent state $|z\rangle = \exp\left[-\frac{1}{2}|z|^2 + za^\dagger\right]|0\rangle$ [1], because for a normally ordered operator function $:\ F(a^\dagger, a):$, where $a^\dagger$, a are the Bose creation, annihilation operators, obeying $[a, a^\dagger] = 1$,

$$\langle z| : F(a^\dagger, a) : |z\rangle = F(z^*, z). \tag{1}$$

A quite powerful method for normally ordering operators is called the technique of integration within an ordered product (IWOP) of operators [2–4], its essential point lies in the fact that Bose operators permute within the normal product symbol : :, i.e.

$$: aa^\dagger := a^\dagger a =: a^\dagger a : . \tag{2}$$

So one can consider the Bose operators which are within : : in an integral as C-number parameters while the integration is performed, for example, the overcompleteness relation of

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the coherent state can be recast into

$$\int \frac{d^2z}{\pi} |z\rangle \langle z| = \int \frac{d^2z}{\pi} e^{-|z|^2+za^\dagger} |0\rangle \langle 0| e^{z^*a} =: \int \frac{d^2z}{\pi} e^{-|z|^2+za^\dagger+z^*a-a^\dagger a} :=: e^{a^\dagger a - a^\dagger a} := 1$$

where we have used

$$|0\rangle \langle 0| =: e^{-a^\dagger a} : . \qquad (3)$$

Using the IWOP technique we can easily deduce many operator identities, for instance, the antinormal product operator $e^{fa^2}e^{ga^{\dagger 2}}$ can be easily re-ordered as a normal ordering, i.e.

$$\begin{aligned} e^{fa^2}e^{ga^{\dagger 2}} &= \int \frac{d^2z}{\pi} e^{fz^2} |z\rangle \langle z| e^{gz^{*2}} =: \int \frac{d^2z}{\pi} e^{-|z|^2+za^\dagger+z^*a+fz^2+gz^{*2}-a^\dagger a} : \\ &= \frac{1}{\sqrt{1-4fg}} \exp(ga^{\dagger 2}/(1-4fg)) \\ &\quad \times \exp[-a^\dagger a \ln(1-4fg)] \exp(fa^2/(1-4fg)). \end{aligned}$$

With the IWOP we can also put the completeness relation of coordinate eigenstates as [5]

$$\int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx\, : e^{-(x-X_1)^2} := 1 \qquad (4)$$

where

$$|x\rangle = \pi^{-\frac{1}{4}} \exp\left\{-\tfrac{1}{2}x^2 + \sqrt{2}xa_1^\dagger - \tfrac{1}{2}a_1^{\dagger 2}\right\} |0\rangle . \qquad (5)$$

For the coordinate operator $X_1 = \frac{1}{\sqrt{2}}(a_1 + a_1^\dagger)$, as a result of (5) we have

$$X_1^n = \int_{-\infty}^{\infty} dx\, x^n\, |x\rangle \langle x| = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx\, x^n : e^{-(x-X_1)^2} : . \qquad (6)$$

Using the mathematical formula

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx\, e^{-\sigma(x-\lambda)^2} x^n = \sqrt{\sigma^{n+1}} \sum_{k=0}^{[n/2]} \frac{n!}{2^{2k}k!(n-2k)!} (\lambda\sigma^{1/2})^{n-2k} \qquad \mathrm{Re}\,\sigma > 0. \qquad (7)$$

we immediatly obtain the normal product form of X_1^n,

$$X_1^n = \sum_{k=0}^{[n/2]} \frac{n!}{2^{2k}k!(n-2k)!} : X_1^{n-2k} : . \qquad (8)$$

One can further prove that the inversion of (8) is

$$\sum_{k=0}^{[n/2]} \frac{n!2^n}{2^{2k}k!(n-2k)!} X_1^{n-2k} = H_n(X_1) = 2^n : X_1^n : . \qquad (9)$$

where H_n is the nth Hermite polynomial. In this work we discuss what is the normally ordered expansion of the Dirac's hermite radial momentum operator [1]

$$\hat{P}_r = \frac{1}{2}\left(\frac{\vec{r}}{r} \cdot \vec{P} + \vec{P} \cdot \frac{\vec{r}}{r}\right) = -i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \qquad \hbar = 1 \qquad (10)$$

where $r = (x^2 + y^2 + z^2)^{1/2}$. To our knowledge, this problem has not been discussed in the literature before. In the following, we shall exploit the IWOP technique to solve this problem. The paper is arranged as follows: In section 2 we first derive the normal product form of the power of radial coordinate operator $\hat{r}^n$, whose definition is shown in equation (11). On the basis of section 2, in section 3 we search for the normal product form of $\hat{P}_r$.

2. The normal product form of the radial coordinate operators $\hat{r}^n$

Enlightened by equation (6), we study the hermite radius operator $\hat{r}^n$, which corresponds to the radius value r^n, via the equation

$$\hat{r}^n = \int \mathrm{d}^3\vec{x}\, |\vec{x}\rangle\langle\vec{x}| r^n \tag{11}$$

where $|\vec{x}\rangle$ is the three dimensional coordinate eigenvector $|\vec{x}\rangle = |x\rangle|y\rangle|z\rangle$. Then what is the normal ordering form of operator $\hat{r}^n$? In another word, how to use the normally ordered power series of a_i and $a_i^\dagger$ to express $\hat{r}^n$? We shall employ the IWOP technique to discuss it. As the first step, we shoud perform the integation of the three-dimensional completeness relation $\int \mathrm{d}^3\vec{x}\, |\vec{x}\rangle\, \langle\vec{x}|$ in the spherical polar coordinate space over the azimuth angles to obtain a one-dimensional radial integral expression.

2.1. One-dimensional radial coordinate integration involved in $\int \mathrm{d}^3\vec{x}\, |\vec{x}\rangle\, \langle\vec{x}|$

Using the IWOP technique we first perform the integration over the azimuth angles in $\int \mathrm{d}^3\vec{x}\, |\vec{x}\rangle\, \langle\vec{x}|$, step by step, in spherical polar coordinate space. Let,

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = \cos\theta \tag{12}$$

then according to (5) we see

$$|\vec{x}\rangle = \pi^{-\frac{3}{4}} \exp\left\{-\frac{1}{2}r^2 + \sqrt{2}r(\sin\theta\cos\varphi a_1^\dagger + \sin\theta\sin\varphi a_2^\dagger + \cos\theta a_3^\dagger) - \frac{1}{2}\sum_{k=1}^{3} a_k^{\dagger 2}\right\}|000\rangle. \tag{13}$$

It then follows from (3) and (13) that the normally ordered form of $|\vec{x}\rangle\langle\vec{x}|$ is

$$\begin{aligned} |\vec{x}\rangle\,\langle\vec{x}| = \pi^{-\frac{3}{2}} :\exp\Bigg\{ & -r^2 + \sqrt{2}r[\sin\theta\cos\varphi(a_1^\dagger + a_1) + \sin\theta\sin\varphi(a_2^\dagger + a_2) \\ & + \cos\theta(a_3^\dagger + a_3)] - \sum_{k=1}^{3}\left[\tfrac{1}{2}\left(a_k^{\dagger 2} + a_k^2\right) + a_k^\dagger a_k\right]\Bigg\}: \\ = \pi^{-\frac{3}{2}} &:\exp\left\{-r^2 + 2r[\sin\theta\cos\varphi X_1 + \sin\theta\sin\varphi X_2 + \cos\theta X_3] - \sum_{k=1}^{3} X_k^2\right\}: \\ = \pi^{-\frac{3}{2}} &:\exp\left\{-r^2 + 2r\vec{n}\cdot\vec{X} - \vec{X}^2\right\}: \end{aligned} \tag{14}$$

where $X_k = \frac{1}{\sqrt{2}}(a_k + a_k^\dagger)$,

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{15}$$

is a unit vector. Note that $\vec{X}$ is now within the normal ordering symbol. Using the Poisson integration formula

$$\begin{aligned} &\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta\, f(m\sin\theta\cos\varphi + n\sin\theta\sin\varphi + k\cos\theta) \\ &\quad = 2\pi\int_{-1}^{1} f\left(u\sqrt{m^2+n^2+k^2}\right)\mathrm{d}u, \end{aligned} \tag{16}$$

and the IWOP technique we can first carry out the integrations $\int \mathrm{d}^3\vec{x}\,|\vec{x}\rangle\langle\vec{x}|$ over the azimuth angles,

$$\begin{aligned}\int \mathrm{d}^3\vec{x}\,|\vec{x}\rangle\langle\vec{x}| &= \pi^{-\frac{3}{2}}\int_0^\infty r^2\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi \sin\theta\,\mathrm{d}\theta : \exp\left\{-r^2+2r\vec{n}\cdot\vec{X}-\vec{X}^2\right\}: \\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty r^2\,\mathrm{d}r\int_{-1}^1 \mathrm{d}u : \exp\left\{-r^2+2ru\sqrt{X_1^2+X_2^2+X_3^2}-\vec{X}^2\right\}: \\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty r^2\,\mathrm{d}r : \frac{1}{2r|\hat{r}|}\exp\{-r^2+2ru|\hat{r}|-|\hat{r}|^2\}|_{-1}^1 : \\ &= :\frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}r\frac{r}{|\hat{r}|}(\mathrm{e}^{-(r-|\hat{r}|)^2}-\mathrm{e}^{-(r+|\hat{r}|)^2}):\end{aligned} \tag{17}$$

where we have defined

$$|\hat{r}| = \left(X_1^2+X_2^2+X_3^2\right)^{1/2}. \tag{18}$$

It is interesting to compare (17), the one-dimensional radial coordinate integration, with the one-dimensional linear coordinate integration (4), the former is more complicated.

2.2. Normal ordering for $\hat{r}^n$

Using (10) and (17) we have

$$\hat{r}^n = \int_0^\infty \frac{\mathrm{d}r}{\sqrt{\pi}} : \frac{r^{n+1}}{|\hat{r}|}\left[\mathrm{e}^{-(r-|\hat{r}|)^2}-\mathrm{e}^{-(r+|\hat{r}|)^2}\right]:. \tag{19}$$

It is easy to understand that the integration in (19) converges iff $n > -3$.

Now keeping in mind that the Bose operators can be considered as C-numbers within : :, we concentrate on the following integrations for $n > -2$,

$$\begin{aligned}I_{n\pm} &\equiv \int_0^\infty \frac{\mathrm{d}r}{\sqrt{\pi}} : \frac{r^{n+1}}{|\hat{r}|}\mathrm{e}^{-(r\pm|\hat{r}|)^2} := \int_{\pm|\hat{r}|}^\infty \frac{\mathrm{d}r}{\sqrt{\pi}} : \frac{(r\mp|\hat{r}|)^{n+1}}{|\hat{r}|}\mathrm{e}^{-r^2} : \\ &= \sum_{k=0}^{n+1} C_{n+1}^k : \left(\int_0^\infty + \int_{\pm|\hat{r}|}^0\right) r^k\frac{\mathrm{d}r}{\sqrt{\pi}|\hat{r}|}(\mp|\hat{r}|)^{n+1-k}\mathrm{e}^{-r^2}:.\end{aligned} \tag{20}$$

(at this point one can see that the case of $n = -2$ should be considered separately.) It then follows from (19) and (20) that

$$\begin{aligned}\hat{r}^n = I_{n-} - I_{n+} &= \sum_{k=0}^{n+1} C_{n+1}^k : |\hat{r}|^{n-k}[1-(-1)^{n+1-k}]\int_0^\infty r^k\frac{\mathrm{d}r}{\sqrt{\pi}}\mathrm{e}^{-r^2} : \\ &\quad + \sum_{k=0}^{n+1} C_{n+1}^k : |\hat{r}|^{n-k}(1+(-1)^{n+1})(-1)^k\int_0^{|\hat{r}|} r^k\frac{\mathrm{d}r}{\sqrt{\pi}}\mathrm{e}^{-r^2}:.\end{aligned} \tag{21}$$

When $n = 2m$, $1+(-1)^{n+1} \equiv 0$, and $1-(-1)^{n+1-k} = 1+(-1)^k \neq 0$ iff k is even, let $k = 2l$, then equation (21) becomes

$$\hat{r}^{2m} = 2\sum_{l=0,1,..}^m C_{2m+1}^{2l} : |\hat{r}|^{2m-2l}\int_0^\infty r^{2l}\frac{\mathrm{d}r}{\sqrt{\pi}}\mathrm{e}^{-r^2} := \sum_{l=0}^m \frac{(2m+1)!}{4^l(2m+1-2l)!l!} : |\hat{r}|^{2m-2l} : \tag{22}$$

which is different from (8). So (20) is a new operator identity. Especially, when $m = 1$, (22) reduces to

$$\hat{r}^2 =: |\hat{r}|^2 : +\tfrac{3}{2}$$

Comparing it with

$$X_1^2+X_2^2+X_3^2 =: \left(X_1^2+X_2^2+X_3^2\right): + \tfrac{3}{2}$$

we see $\hat{r}^2 = X_1^2+X_2^2+X_3^2$, thus (22) can also be written as

$$\left(X_1^2+X_2^2+X_3^2\right)^m = \sum_{k=0}^{m} \frac{(2m+1)!}{4^k(2m+1-2k)!k!} : \left(X_1^2+X_2^2+X_3^2\right)^{m-k} : \qquad (23)$$

When $m = 2, 3$, the first part of (22) reduces to

$$\hat{r}^4 =: |\hat{r}|^4 : + : 5|\hat{r}|^2 : + \frac{15}{4}$$

$$\hat{r}^6 =: |\hat{r}|^6 : + : \frac{21}{2}|\hat{r}|^4 : + : \frac{105}{4}|\hat{r}|^2 : + \frac{105}{8}$$

respectively. Now we turn to the $n = -2$ case,

$$\begin{aligned}
\hat{r}^{-2} &= \int_0^\infty \frac{\mathrm{d}r}{\sqrt{\pi}} : \frac{r^{-1}}{|\hat{r}|}\left[\mathrm{e}^{-(r-|\hat{r}|)^2} - \mathrm{e}^{-(r+|\hat{r}|)^2}\right] := \int_0^\infty \frac{\mathrm{d}r}{\sqrt{\pi}} : \frac{1}{r|\hat{r}|}(\mathrm{e}^{2r|\hat{r}|} - \mathrm{e}^{-2r|\hat{r}|})\mathrm{e}^{-r^2-|\hat{r}|^2} : \\
&= \sum_{k=0}^\infty \int_0^\infty \frac{\mathrm{d}r}{\sqrt{\pi}} : \frac{2(2r|r|)^{2k+1}}{(2k+1)!r|\hat{r}|}\mathrm{e}^{-r^2-|\hat{r}|^2} \\
&:= \sum_{k=0}^\infty : \frac{2^{2k+1}|\hat{r}|^{2k}}{(2k+1)!}\mathrm{e}^{-|\hat{r}|^2} : \int_0^\infty \frac{2r^{2k}}{\sqrt{\pi}}\mathrm{e}^{-r^2}\mathrm{d}r \\
&= \sum_{k=0}^\infty : \frac{2^{2k+1}|\hat{r}|^{2k}}{(2k+1)!}\mathrm{e}^{-|\hat{r}|^2}\frac{(2k)!}{2^{2k}k!} \\
&:= \sum_{k=0}^\infty : \frac{2|\hat{r}|^{2k}}{(2k+1)k!}\sum_{j=0}^\infty \frac{(-|\hat{r}|^2)^j}{j!} : \\
&= 2\sum_{k=0}^\infty \sum_{j=0}^k \frac{(-1)^{k-j}}{j!(k-j)!(2j+1)} : |\hat{r}|^{2k} : .
\end{aligned}$$

Using the formula of Beta functions

$$B\left(k+1,\frac{1}{2}\right) = 2\int_0^1 (1-t^2)^k \mathrm{d}t = 2\int_0^1 \mathrm{d}t \sum_{j=0}^k \frac{k!(-1)^j t^{2j}}{j!(k-j)!} = 2\sum_{j=0}^k \frac{k!(-1)^j}{j!(k-j)!(2j+1)}$$

and $B(k+1,\frac{1}{2}) = \Gamma\left(\frac{1}{2}\right)\Gamma\left(k+1\right)/\Gamma\left(k+\frac{3}{2}\right)$ we get

$$\hat{r}^{-2} = \sum_{k=0}^\infty \frac{(-1)^k}{k!}B\left(k+1,\frac{1}{2}\right) : \hat{r}^{2k} := \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1}k!}{(2k+1)!} : \hat{r}^{2k} : . \qquad (24)$$

Further, from equation (21) we see that when $n = 2m-1$ is odd, $1-(-1)^{n+1-k} = 1-(-1)^k \neq 0$ iff k is odd, and $1+(-1)^{n+1} \equiv 2$, so (21) leads to

$$\begin{aligned}
\hat{r}^{2m-1} &= \sum_{p=0}^{m-1} C_{2m}^{2p+1} : |\hat{r}|^{2m-2p-2}\int_0^\infty 2r^{2p+1}\frac{\mathrm{d}r}{\sqrt{\pi}}\mathrm{e}^{-r^2} : + 2 : \int_0^{|\hat{r}|}(r-|\hat{r}|)^{2m}\frac{\mathrm{d}r}{\sqrt{\pi}|\hat{r}|}\mathrm{e}^{-r^2} : \\
&= \sum_{p=0}^{m-1} : C_{2m}^{2p+1}|\hat{r}|^{2m-2p-2}p!/\sqrt{\pi} : + 2 : \int_0^{|\hat{r}|}(r-|\hat{r}|)^{2m}\frac{\mathrm{d}r}{\sqrt{\pi}|\hat{r}|}\mathrm{e}^{-r^2} : . \qquad (25)
\end{aligned}$$

Expanding the factor e^{-r^2} into infinity series, we can complete the integration

$$: \frac{2}{|\hat{r}|}\int_0^{|\hat{r}|}(r-|\hat{r}|)^{2m}e^{-r^2}dr :=: \frac{2}{|\hat{r}|}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^{|\hat{r}|}(r-|\hat{r}|)^{2m}r^{2k}dr :$$

$$=: \frac{2}{|\hat{r}|}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}|\hat{r}|^{2m+2k+1}\int_0^1(r-1)^{2m}r^{2k}dr :$$

$$=: \frac{2}{|\hat{r}|}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}|\hat{r}|^{2m+2k+1}B(2m+1,2k+1) :$$

$$= 2:\sum_{k=0}^{\infty}\frac{(-1)^k(2m)!(2k)!}{k!(2m+2k+1)!}|\hat{r}|^{2(m+k)} :, \tag{26}$$

where $B(p,q)$ is the well-known Beta function. Substituting (25) into (24) we finally obtain the normally ordered expansion of r^{2m-1}:

$$\hat{r}^{2m-1} =: \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k(2m)!(2k)!}{k!(2m+2k+1)!}|\hat{r}|^{2m+2k} : + \frac{1}{\sqrt{\pi}}\sum_{k=0}^{m-1} : k!C_{2m}^{2k+1}|\hat{r}|^{2m-2k-2} : . \tag{27}$$

For example, when $m=1$

$$\hat{r} = 2/\sqrt{\pi} + : \frac{4}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{k!(2k+3)!}|\hat{r}|^{2k+2} :$$

when $m=0$

$$\hat{r}^{-1} = \int_0^{\infty}\frac{dr}{\sqrt{\pi}} : \frac{1}{|\hat{r}|}[e^{-(r-|\hat{r}|)^2} - e^{-(r+|\hat{r}|)^2}] :=: \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(2k+1)}|\hat{r}|^{2k} : . \tag{28}$$

3. The normally ordered expansion of $\hat{P}_r$

Based on the above results we are able to derive the normally ordered expansion of $\hat{P}_r$. Using (14) and (10) we have

$$\hat{P}_r = \int d^3\vec{x}\ \hat{P}_r\,|\vec{x}\rangle\langle\vec{x}| = \tfrac{1}{2}\left(\int d^3\vec{x}\ \hat{P}_r\,|\vec{x}\rangle\langle\vec{x}| + \int d^3\vec{x}\,|\vec{x}\rangle\langle\vec{x}|\ \hat{P}_r\right)$$

$$= i\tfrac{1}{2}\left[\int d^3\vec{x}\left(-r+\sqrt{2}\vec{n}\cdot\vec{a}^{\dagger}+\frac{1}{r}\right)|\vec{x}\rangle\langle\vec{x}| - \int d^3\vec{x}\,|\vec{x}\rangle\langle\vec{x}|\left(-r+\sqrt{2}\vec{n}\cdot\vec{a}+\frac{1}{r}\right)\right]$$

$$= \pi^{-\frac{3}{2}}\int d^3\vec{x} : \vec{n}\cdot\vec{P}\exp\{-r^2+2r\vec{n}\cdot\vec{X}-\vec{X}^2\} : . \tag{29}$$

Since every vector can be decomposed as a sum of its longitudinal and transversal components, with respect to $\vec{X}$ ($\vec{X}$ is considered as a fictitious vector) we can decompose $\vec{n}$ as

$$\vec{n} = \frac{\vec{n}\cdot\vec{X}}{\vec{X}^2}\vec{X} + \frac{\vec{X}\times(\vec{n}\times\vec{X})}{\vec{X}^2}. \tag{30}$$

Substituting (29) into (28) we have

$$\hat{P}_r = \pi^{-\frac{3}{2}}\int d^3\vec{x} : \vec{n}\exp\{-r^2+2r\vec{n}\cdot\vec{X}-\vec{X}^2\}\cdot\vec{P} :$$

$$= \pi^{-\frac{3}{2}}\int d^3\vec{x} : \left(\frac{\vec{n}\cdot\vec{X}}{\vec{X}^2}\vec{X}+\frac{\vec{X}\times(\vec{n}\times\vec{X})}{\vec{X}^2}\right)\exp\{-r^2+2r\vec{n}\cdot\vec{X}-\vec{X}^2\}\cdot\vec{P} : .$$

Taking the symmetric property into consideration, the contribution of the transversal component to the integration is zero (for a more detailed explanation, see the appendix), in another word, we can say that the integration result of the vector $\int \mathrm{d}^3\vec{x} : \vec{n}\exp\{-r^2 + 2r\vec{n}\cdot\vec{X} - \vec{X}^2\}:$ will be parallel with $\vec{X}$. So we have

$$
\begin{aligned}
\hat{P}_r &= \pi^{-\frac{3}{2}} \int_0^\infty r^2\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \sin\theta\,\mathrm{d}\theta : \vec{n}\cdot\vec{X}\exp\{-r^2 + 2r\vec{n}\cdot\vec{X} - \vec{X}^2\}\vec{X}\cdot\vec{P}/|\hat{r}|^2 : \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty r^2\,\mathrm{d}r \int_{-1}^1 \mathrm{d}u : u\sqrt{X_1^2 + X_2^2 + X_3^2} \\
&\quad \times \exp\left\{-r^2 + 2ru\sqrt{X_1^2 + X_2^2 + X_3^2} - \vec{X}^2\right\}\vec{X}\cdot\vec{P}/|\hat{r}|^2 : \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty r^2\,\mathrm{d}r : \frac{1}{(2r)^2|\hat{r}|}(2ru|\hat{r}| - 1)\exp\{-r^2 + 2ru|\hat{r}| - |\hat{r}|^2\}|_{-1}^1 \vec{X}\cdot\vec{P}/|\hat{r}|^2 : \\
&=: \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{d}r \frac{1}{2|\hat{r}|}\left[(2r|\hat{r}| - 1)\mathrm{e}^{-(r-|\hat{r}|)^2} - (-2r|\hat{r}| - 1)\mathrm{e}^{-(r+|\hat{r}|)^2}\right]\vec{X}\cdot\vec{P}/|\hat{r}|^2 : \\
&=: \frac{1}{\sqrt{\pi}} \left[\int_0^\infty \mathrm{d}r \frac{1}{2|\hat{r}|} 2r|\hat{r}|(\mathrm{e}^{-(r-|\hat{r}|)^2} + \mathrm{e}^{-(r+|\hat{r}|)^2})\right. \\
&\quad \left. - \int_0^\infty \mathrm{d}r \frac{1}{2|\hat{r}|}(\mathrm{e}^{-(r-|\hat{r}|)^2} - \mathrm{e}^{-(r+|\hat{r}|)^2})\right]\vec{X}\cdot\vec{P}/|\hat{r}|^2 :\equiv W_1 + W_2
\end{aligned} \tag{31}
$$

where the first integral can be proceeded by analogy with deriving (19),

$$
\begin{aligned}
W_1 &=: \frac{1}{\sqrt{\pi}} \left[\int_{-|\hat{r}|}^\infty (r + |\hat{r}|)\,\mathrm{d}r\,\mathrm{e}^{-r^2} + \int_{|\hat{r}|}^\infty (r - |\hat{r}|)\,\mathrm{d}r\,\mathrm{e}^{-r^2}\right]\vec{X}\cdot\vec{P}/|\hat{r}|^2 : \\
&=: \frac{1}{\sqrt{\pi}} \left[\left(\int_{-|\hat{r}|}^\infty + \int_{|\hat{r}|}^\infty\right) r\,\mathrm{d}r\,\mathrm{e}^{-r^2} + |\hat{r}|\left(\int_{-|\hat{r}|}^\infty - \int_{|\hat{r}|}^\infty\right)\mathrm{d}r\,\mathrm{e}^{-r^2}\right]\vec{X}\cdot\vec{P}/|\hat{r}|^2 : \\
&=: \frac{1}{\sqrt{\pi}} \left[\mathrm{e}^{-|\hat{r}|^2} + 2\sum_{k=0}^\infty \frac{(-1)^k|\hat{r}|^{2k+2}}{(2k+1)k!}\right]\vec{X}\cdot\vec{P}/|\hat{r}|^2 :
\end{aligned} \tag{32}
$$

while the second integral in (30) corresponds to (27), thus

$$
\begin{aligned}
\hat{P}_r &=: \frac{1}{\sqrt{\pi}} \left[\left(\mathrm{e}^{-|\hat{r}|^2} + 2\sum_{k=0}^\infty \frac{(-1)^k|\hat{r}|^{2k+2}}{(2k+1)k!}\right) - \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)k!}|\hat{r}|^{2k}\right]\vec{X}\cdot\vec{P}/|\hat{r}|^2 : \\
&=: \sum_{k=1}^\infty \frac{4k(-1)^{k-1}|\hat{r}|^{2k-2}}{(4k^2-1)k!\sqrt{\pi}}\vec{X}\cdot\vec{P} :=: \sum_{k=0}^\infty \frac{4(-1)^k|\hat{r}|^{2k}}{(4k^2+8k+3)k!\sqrt{\pi}}\vec{X}\cdot\vec{P} : .
\end{aligned} \tag{33}
$$

Before we end this paper we briefly mention some application of the above new identities, for example, from equation (22), we can directly write down the coherent matrix element of $\hat{r}^{2m}$

$$
\begin{aligned}
&\langle z_1, z_2, z_3|\,\hat{r}^{2m}\,|z_1', z_2', z_3'\rangle \\
&\quad = \sum_{k=0}^m \frac{(2m+1)!}{2^{k+m}(2m+1-2k)!k!}\left[\sum_{j=1}^3 (z_j^* + z_j')^2\right]^{m-k}\langle z_1, z_2, z_3|z_1', z_2', z_3'\rangle.
\end{aligned} \tag{34}
$$

As its application we consider a three-dimensional isotropic oscillator perturbed by a Hamiltonian $V = \lambda r^{2m}$, where λ is small enough. Then to the first order of perturbation, the energy correction to the unpertubed ground state $|000\rangle = |z_1 = 0, z_2 = 0, z_3 = 0\rangle$ is

$$
\langle 000|\,\lambda\hat{r}^{2m}\,|000\rangle = \lambda\frac{(2m+1)!}{2^{2m}m!} \tag{35}
$$

which shows that, no matter how small the λ is, the pertubation may fail when m is large enough.

In summary, we have derived the normally ordered expansions of the power of radius operators and based on which we have further obtained the normal product form of the Dirac's radial momentum operator. The work once again demonstrates the usefulness of the IWOP technique.

Appendix

Here we prove that the second term in equation (29) contributes nothing to the integration in equation (28).We need to demonstrate that

$$\int \mathrm{d}^3\vec{x} : (\vec{n} \times \vec{X}) \exp\{-r^2 + 2r\vec{n} \cdot \vec{X} - \vec{X}^2\} := 0. \tag{36}$$

By noting that

$$\vec{n} \times \vec{X} = \begin{pmatrix} i & j & k \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ X_1 & X_2 & X_3 \end{pmatrix}$$

so equation (36) is equivalent to

$$\int \mathrm{d}^3\vec{x} : (X_3\sin\theta\sin\varphi - X_2\cos\theta)\exp\{-r^2 + 2r\vec{n}\cdot\vec{X} - \vec{X}^2\} := 0$$
$$\int \mathrm{d}^3\vec{x} : (X_3\sin\theta\cos\varphi - X_1\cos\theta)\exp\{-r^2 + 2r\vec{n}\cdot\vec{X} - \vec{X}^2\} := 0$$
$$\int \mathrm{d}^3\vec{x} : (X_2\sin\theta\cos\varphi - X_1\sin\theta\sin\varphi)\exp\{-r^2 + 2r\vec{n}\cdot\vec{X} - \vec{X}^2\} := 0.$$

Writing $z = \mathrm{e}^{i\varphi}$, and

$$\begin{aligned} : \exp(2r\vec{n}\cdot\vec{X}) :&=: \exp\left\{2r\left(\sin\theta\frac{\mathrm{e}^{\mathrm{i}\varphi}+\mathrm{e}^{-\mathrm{i}\varphi}}{2}X_1 + \sin\theta\frac{\mathrm{e}^{\mathrm{i}\varphi}-\mathrm{e}^{-\mathrm{i}\varphi}}{2\mathrm{i}}X_2 + \cos\theta X_3\right)\right\} : \\ &=: \exp\left\{2r\left(\frac{\sin\theta(X_1-\mathrm{i}X_2)}{2}z + \frac{\sin\theta(X_1+\mathrm{i}X_2)}{2z} + \cos\theta X_3\right)\right\} : \\ &=: \exp\left(\lambda z + \frac{\mu}{z}\right)\exp(2rX_3\cos\theta) : \end{aligned}$$

where

$$\lambda = r\sin\theta(X_1 - \mathrm{i}X_2)$$
$$\mu = r\sin\theta(X_1 + \mathrm{i}X_2).$$

we want to calculate (for brevity we omit the symbol : :)

$$\int_0^\pi \sin\theta \int_0^{2\pi} \vec{n}\exp(2r\vec{n}\cdot\vec{X})\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{37}$$

Its three components are

$$N_1 = \int_0^\pi \sin\theta \int_0^{2\pi} \sin\theta\cos\varphi\exp(2r\vec{n}\cdot\vec{X})\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$N_2 = \int_0^\pi \sin\theta \int_0^{2\pi} \sin\theta\sin\varphi\exp(2r\vec{n}\cdot\vec{X})\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$N_3 = \int_0^\pi \sin\theta \int_0^{2\pi} \cos\theta\exp(2r\vec{n}\cdot\vec{X})\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

in which

$$
\begin{aligned}
&\int_0^{2\pi} \cos\varphi \exp(2r\vec{n}\cdot\vec{X})\,\mathrm{d}\varphi \\
&\qquad = \oint \frac{1}{2\mathrm{i}}\mathrm{d}\left(z-\frac{1}{z}\right)\exp\left(\lambda z+\frac{\mu}{z}\right)\exp\left(2rX_3\cos\theta\right) \\
&\qquad = \pi\exp(2rX_3\cos\theta)\sum_{k=0}^{\infty}\frac{C_{2k+1}^{k}}{(2k+1)!}[\lambda^k\mu^{k+1}+\lambda^{k+1}\mu^k] \\
&\qquad = X_1\cdot 2\pi r\sin\theta\exp(2rX_3\cos\theta)\sum_{k=0}^{\infty}\frac{\lambda^k\mu^k}{(k+1)!k!}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{2\pi}\sin\varphi\exp(2r\vec{n}\cdot\vec{X})\,\mathrm{d}\varphi = X_2\cdot 2\pi r\sin\theta X_2\exp(2rX_3\cos\theta)\sum_{k=0}^{\infty}\frac{\lambda^k\mu^k}{(k+1)!k!} \\
&\int_0^{2\pi}\exp(2r\vec{n}\cdot\vec{X})\,\mathrm{d}\varphi = 2\pi\exp(2rX_3\cos\theta)\sum_{k=0}^{\infty}\frac{\lambda^k\mu^k}{(k!)^2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
N_1 &= X_1\cdot W_1 \\
N_2 &= X_2\cdot W_1 \\
N_3 &= W_2
\end{aligned}
$$

where

$$
\begin{aligned}
W_1 &= \int_0^{\pi}\sin\theta\,\mathrm{d}\theta\left\{\sin\theta\,2\pi r\ \sin\theta\exp(2rX_3\cos\theta)\sum_{k=0}^{\infty}\frac{\lambda^k\mu^k}{(k+1)!k!}\right\} \\
W_2 &= \int_0^{\pi}\sin\theta\,\mathrm{d}\theta\left\{\cos\theta\,2\pi\ \exp(2rX_3\cos\theta)\sum_{k=0}^{\infty}\frac{\lambda^k\mu^k}{(k!)^2}\right\}.
\end{aligned}
$$

be we only need to prove that $X_3\cdot W_1 = W_2$

$$
\begin{aligned}
W_1 &= \sum_{k=0}^{\infty}\frac{2\pi r\left[r^2\left(X_1^2+X_2^2\right)\right]^k}{(k+1)!k!}\int_0^{\pi}\sin^{2k+3}\theta\,\mathrm{d}\theta\exp(2rX_3\cos\theta) \\
&= \sum_{k=0}^{\infty}\frac{2\pi r\left[r^2\left(X_1^2+X_2^2\right)\right]^k}{(k+1)!k!}\sum_{j=0}^{\infty}\frac{(2rX_3)^{2j}}{(2j)!}\int_0^{\pi}\sin^{2k+3}\theta\cos^{2j}\theta\,\mathrm{d}\theta \\
&= \sum_{k=0}^{\infty}\frac{2\pi r\left[r^2\left(X_1^2+X_2^2\right)\right]^k}{(k+1)!k!}\sum_{j=0}^{\infty}\frac{(2rX_3)^{2j}}{(2j)!}\frac{2^{k+2}(k+1)!(2j-1)!!}{(2k+2j+3)!!} \\
&= \sum_{k=0}^{\infty}\frac{2^{k+3}\pi r\left[r^2\left(X_1^2+X_2^2\right)\right]^k}{k!}\sum_{j=0}^{\infty}\frac{(2rX_3)^{2j}}{(2j)!!(2k+2j+3)!!}
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= \int_0^{\pi} \sin\theta \, d\theta \left\{ \cos\theta 2\pi \exp(2rX_3\cos\theta) \sum_{k=0}^{\infty} \frac{\lambda^k \mu^k}{(k!)^2} \right\} \\
&= \sum_{k=0}^{\infty} \frac{2\pi \left[r^2 \left(X_1^2 + X_2^2\right)\right]^k}{(k!)^2} \int_0^{\pi} \sin^{2k+1}\theta \, \cos\theta \, d\theta \exp(2rX_3\cos\theta) \\
&= \sum_{k=0}^{\infty} \frac{2\pi \left[r^2 \left(X_1^2 + X_2^2\right)\right]^k}{(k!)^2} \sum_{j=0}^{\infty} \frac{(2rX_3)^{2j+1}}{(2j+1)!} \int_0^{\pi} \sin^{2k+1}\theta \cos^{2j+2}\theta \, d\theta \\
&= \sum_{k=0}^{\infty} \frac{2\pi \left[r^2 \left(X_1^2 + X_2^2\right)\right]^k}{(k!)^2} \sum_{j=0}^{\infty} \frac{(2rX_3)^{2j+1}}{(2j+1)!} \frac{2^{k+1}k!(2j+1)!!}{(2k+2j+3)!!} \\
&= \sum_{k=0}^{\infty} \frac{X_3 \cdot 2^{k+3}\pi r \left[r^2 \left(X_1^2 + X_2^2\right)\right]^k}{k!} \sum_{j=0}^{\infty} \frac{(2rX_3)^{2j}}{(2j)!!(2k+2j+3)!!} \\
&= X_3 \cdot W_1.
\end{aligned}
$$

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